CCFU Proof 12

Geodesic Translation Length $\ell(A_4^2) = 2\ln\varphi$

Given. Let $\varphi = (1+\sqrt{5})/2$, so $\varphi^2 = \varphi + 1$.

$$\mathrm{spec}(A_4^2) = \{\varphi^2,\ 1/\varphi^2,\ 1,\ 1\}, \qquad A_4^2 \in SO^+(3,1), \qquad \det(A_4^2) = 1.$$

[Dependency: Proof 4]

Step 1 — Compute $\varphi^2 + 1/\varphi^2$.
Using $\varphi^2 = \varphi + 1$:

$$\varphi^4 = (\varphi+1)^2 = \varphi^2 + 2\varphi + 1 = (\varphi+1) + 2\varphi + 1 = 3\varphi + 2,$$

$$\varphi^4 + 1 = 3\varphi + 3 = 3(\varphi+1) = 3\varphi^2,$$

$$\varphi^2 + 1/\varphi^2 = (\varphi^4+1)/\varphi^2 = 3\varphi^2/\varphi^2 = 3. \qquad \blacksquare$$

Step 2 — Trace.

$$\mathrm{tr}(A_4^2) = \varphi^2 + 1/\varphi^2 + 1 + 1 = 3 + 2 = 5. \qquad \blacksquare$$

Step 3 — Translation length formula. For a pure hyperbolic isometry in $SO^+(3,1)$ with eigenvalues $\{e^\ell, e^{-\ell}, 1, 1\}$:

$$\cosh(\ell) = \frac{\mathrm{tr}(A_4^2) - 2}{2} = \frac{5-2}{2} = \frac{3}{2}. \qquad \blacksquare$$

Step 4 — Verification.

$$\cosh(2\ln\varphi) = \frac{\varphi^2 + 1/\varphi^2}{2} = \frac{3}{2}. \qquad \checkmark$$

Therefore $\ell(A_4^2) = 2\ln\varphi$. $\blacksquare$

Corollary. One step of B (the boost factor of A_4, Proof 11) translates along its geodesic axis by $\ln\varphi$.